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Parameter Estimation of Probability Distributions Using Bernstein and Rational Bernstein Polynomial-Based Approaches

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Bernstein ve Rasyonel Bernstein Polinom Tabanlı Yaklaşımlar Kullanılarak Olasılık Dağılımlarının Parametre Tahmini

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Abstract

This study examines two different approaches based on Bernstein polynomials and rational Bernstein polynomials for parameter estimation of probability distributions. It is discussed how both methods can be used in the parameter estimation process, and it is aimed to determine the optimal parameters with the least squares method. Monte Carlo simulations are performed to evaluate the effectiveness of the methods, and their estimation performances are analyzed for various distributions. Simulation results demonstrate that rational Bernstein polynomials achieve lower mean squared error values, which consequently raise parameter estimation accuracy through enhanced flexibility.

Keywords: *Bernstein Polynomials; Rational Bernstein Polynomials; Parameter Estimation; Nonparametric Estimation; Mean Squared Error*

Öz

Bu çalışmada olasılık dağılımlarının parametre kestirimi için Bernstein polinomları ve rasyonel Bernstein polinomlarına dayalı iki farklı yaklaşım incelenmektedir. Her iki yöntemin parametre kestirim sürecinde nasıl kullanılabileceği tartışılmakta ve en küçük kareler yöntemi ile optimum parametrelerin belirlenmesi amaçlanmaktadır. Yöntemlerin etkinliğini değerlendirmek için Monte Carlo simülasyonları gerçekleştirilmekte ve çeşitli dağılımlar için kestirim performansları analiz edilmektedir. Simülasyon sonuçları rasyonel Bernstein polinomlarının daha düşük ortalama karesel hata değerlerine ulaştığını ve bunun sonucunda artan esneklik yoluyla parametre kestirim doğruluğunu artırdığını göstermektedir.

Anahtar Kelimeler: *Bernstein polinomları; Rasyonel Bernstein polinomları; Parametre kestirimi; Parametrik olmayan kestirim; Ortalama karesel hata*

1. Introduction

Parameter estimation is a fundamental aspect of statistical data analysis and nonparametric modeling. Estimating the parameters of a distribution—such as the mean and variance—that define its essential characteristics is crucial before conducting any statistical analysis. Accurate parameter estimates in probability and statistics enable researchers to develop strong statistical models and gain a deeper understanding of distributional properties. Various disciplines, including biology, engineering, economics, and the social sciences, widely apply these estimates, particularly in modeling problems and probabilistic analyses.

The current study aims to estimate parameters using rational Bernstein polynomials and classical Bernstein polynomials. The literature widely recognizes equations based on Bernstein polynomials and their generalizations as fundamental tools for function approximation. When viewed from a probabilistic perspective, these commonly used polynomial methods facilitate the estimation of probability density and distribution functions. Their simplicity in approximating functions across many intervals makes them particularly appealing. Bernstein polynomials were originally introduced by Bernstein (1912) as a constructive proof of the Weierstrass Approximation Theorem. Later, Lorentz (1986) provided a comprehensive mathematical foundation for these polynomials, emphasizing their uniform convergence and approximation capabilities. Vitale (1975) was the first to propose a nonparametric technique for estimating density functions in probability and statistics using Bernstein polynomials. Petrone (1999) explored the use of Bernstein polynomials from a Bayesian perspective for probability density estimation. This line of research was further advanced by Petrone and Wasserman (2002), who examined the asymptotic properties of Bernstein-based estimators and demonstrated their consistency and flexibility within Bayesian frameworks. Ghosal (2001) also contributed significantly by investigating convergence rates in density estimation using Bernstein polynomials. Babu et al. (2002) discussed the use of Bernstein

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polynomials for constructing smooth density and distribution functions. Their findings revealed that this method outperforms traditional kernel density estimators. Kakizawa (2004) considered Bernstein polynomials as nonparametric priors for continuous probability densities. Babu and Chaubey (2006) employed these polynomials to estimate the smooth density function of dependent random vectors and proposed a method suitable for high-dimensional data. Turnbull and Ghosh (2014) demonstrated that their method using Bernstein polynomials for unimodal density estimation outperformed existing techniques. Additionally, Carnicero et al. (2018) applied Bernstein polynomial-based estimators to circular data, highlighting their flexibility across diverse data types. In a more recent study, Erdoğan et al. (2019) introduced a novel approach for estimating the cumulative distribution function using rational Bernstein polynomials. Their results indicated that rational Bernstein polynomials offer greater flexibility and, consequently, more accurate approximations.

This work investigates the influence of two approaches for estimating the cumulative distribution function (CDF) on parameter estimation. Without compromising the advantages of classic Bernstein polynomials, rational Bernstein polynomials provide additional control and flexibility for better model fitting. By incorporating extra weight factors, rational Bernstein polynomials offer greater modeling flexibility than classical Bernstein polynomials, allowing for more precise function fitting and, consequently, improved model accuracy. Using CDF estimates generated via rational and classical Bernstein polynomials, the optimal parameter estimates are determined based on the least squares approach. A Monte Carlo simulation is employed to assess the performance of both techniques; the mean squared error (MSE) is used to guide the comparison of the results.

The remainder of this work is organized as follows. Section 2 presents the mathematical foundations of Bernstein and rational Bernstein estimators. Section 3 introduces the parameter estimation method using the least-squares approach. Section 4 reports the results of the Monte Carlo simulation studies and compares the parameter estimates obtained from the Bernstein and rational Bernstein polynomials. Section 5 evaluates the effectiveness of both methods in parameter estimation using a real data set. Section 6 concludes the study.

2. Bernstein Polynomials and Cumulative distribution function estimation

Estimating cumulative distribution functions is a fundamental problem in probability theory and statistics. Nonparametric methods stand out in these estimations due to their flexibility and lack of reliance on distributional assumptions; one of these methods is the Bernstein polynomials. Bernstein polynomials can smoothly approximate any continuous function. In statistical analysis, rational generalizations of these polynomials are also employed to better fit the data and achieve lower estimation errors. We mathematically demonstrate two different distribution function estimation methods, one based on the Bernstein polynomial proposed by Babu et al. (2002), and the other based on the rational Bernstein polynomial proposed by Erdoğan et al. (2019), and summarize the advantages of these methods.

2.1 CDF estimation with classical Bernstein polynomials

Bernstein polynomials are extensively utilized in mathematical literature. Furthermore, these polynomials have begun to be employed in the realm of distribution functions and density estimates, supported by the probabilistic demonstration of the Weierstrass Approximation Theorem published by Bernstein (1912). This theorem states that Bernstein polynomials converge uniformly to any continuous function, $f(x)$, on the closed interval $[a, b]$ as the degree of $m \to \infty$, specifically for $a < x < b$. The Bernstein-Weierstrass Approximation theorem ensures that when the degree of the polynomial approaches infinity, the Bernstein polynomial approximation converges uniformly to the target function. In a more basic scenario, we can examine the case where a random variable takes values in the interval $[0, 1]$. The approximation of a continuous function $F(x)$ using Bernstein polynomials is expressed as follows (Lorentz, 1986):

$$B_m(F)(x) = \sum_{k=0}^{m} F\left(\frac{k}{m}\right) B_{k,m}(x) \qquad (1)$$

where m is the degree of the polynomial, $F\left(\frac{k}{m}\right)$ is the value of the function $F(x)$ at the point k/m, $B_{k,m}(x)$ are called Bernstein basis functions, and they are given by:

$$B_{k,m}(x) = \binom{m}{k} x^k (1-x)^{m-k}, x \in [0,1] \qquad (2)$$

where $\binom{m}{k}$ represent the binomial coefficient. This methodology enables the smooth estimation of the function $F(x)$ by Bernstein polynomials. Babu et al. (2002) proposed a technique for estimating the cumulative distribution function (CDF) using these polynomials. Classical Bernstein polynomials yield smooth approximations of the actual distribution function as follows:

$$\hat{F}_{m,n}(x) = \sum_{k=0}^{m} F_n\left(\frac{k}{m}\right) B_{k,m}(x) \qquad (3)$$

where $F_n(x) = n^{-1} \sum_{i=1}^{n} I\{X_i \le x\}$ is the empirical cumulative distribution function estimator and n denotes

the sample size. This method yields smoother approximations than the empirical distribution function and produces a monotonically increasing function. Nonetheless, a drawback is that Bernstein polynomials are prone to errors, particularly at extreme points in low-degree scenarios.

2.2 CDF estimation with rational Bernstein polynomials

It is possible to generalize Bernstein polynomials in a variety of ways (Oruç, Phillips, and Davis, 1999; Barry, Beatty, and Goldman, 1992; Budakçı and Oruç, 2012). One of the most striking of them is the rational Bernstein polynomials. A strong substitute for calculating the distribution function, rational Bernstein polynomials are now employed in probability and statistics. The rational Bernstein polynomial-based distribution function estimator by Erdoğan et al. (2019) is a more adaptable extension of the classical Bernstein polynomials that Babu et al. (2002) suggested. Using weighted basis functions, this method approximates the cumulative distribution function. Because rational Bernstein polynomials generalize Bernstein polynomials with extra weight coefficients, they offer a more sensitive function approach. Using rational Bernstein polynomials, the cumulative distribution function is estimated as follows:

$$\hat{F}_{m,n,w}(x) = \sum_{k=0}^{m} F_n\left(\frac{k}{m}\right) R_{k,m,w}(x) \qquad (4)$$

where,

$$R_{k,m,w}(x) = \frac{w_k \binom{m}{k} x^k (1-x)^{m-k}}{\sum_{k=0}^{m} w_k \binom{m}{k} x^k (1-x)^{m-k}}, \qquad x \in [0,1]$$

(Erdoğan et al., 2019). The weights w_k are preferred to manipulate the curve's shape, in contrast to the fixed structure of classical Bernstein polynomials. Optimization techniques determine the ideal weights by minimizing the difference between the rational estimator and the empirical distribution function.

The following optimization problem is considered to obtain the w_k coefficients.

$$\hat{w} = \operatorname{argmin} \sum_{i=1}^{n} \left[F_n(x_i) - \hat{F}_{m,n,w}(x)\right]^2 \qquad (5)$$

$F_n(x)$ represents the empirical distribution function. When weights are set based on the data structure, there are several benefits compared to classical Bernstein polynomials. These include lower MSE values, more modeling options, better results in small samples, and better results at extreme values.

3. Parameter estimation

The parameters of a distribution are any quantity of a statistical population that summarizes or describes an aspect of the population. Parameters determine characteristics of a distribution, such as the mean and variance, as well as critical features such as the shape and scale of the distribution. We can estimate the parameters of a distribution using estimates of distribution functions based on both Bernstein and rational Bernstein polynomials. This work calculates the optimal parameter values with the help of the least squares method for both estimators. In this section, we explain how to estimate the parameters of distributions using classical Bernstein and rational Bernstein polynomials.

3.1 Optimization with Least Squares Method

We aim to approximate the parameters of the distribution function using the estimates of the distribution functions obtained through classic Bernstein and rational Bernstein polynomials. For this purpose, we utilize the least squares approach to minimize the error function, thereby determining the optimal parameters (Kutner et al., 2005).

$$\theta^* = \operatorname{argmin} \sum_{i=1}^{n} \left[\hat{F}(X_i) - F(X_i, \theta)\right]^2 \qquad (6)$$

The distribution function $\hat{F}(X_i)$ is estimated using Bernstein or rational Bernstein polynomials, and the parameter vector is denoted by θ. This optimization technique yields θ values that guarantee the estimated distribution function best matches the actual distribution function. This optimization procedure is carried out using the optim function in R, employing the L-BFGS-B algorithm.

We evaluate the estimated model's correctness using the mean squared error measure:

$$MSE = \frac{1}{N} \sum_{i=1}^{N} [\theta_i^* - \theta)]^2 \qquad (7)$$

where θ represents the actual parameter values, N is the number of Monte Carlo simulation replications, and θ^* indicates the estimated parameter values. A lower MSE value indicates that the estimated parameter better approximates the true parameter.

4. Simulation studies

This section designs a comprehensive Monte Carlo simulation study to measure the effect of rational Bernstein and Bernstein polynomials on parameter estimation. Six different distributions are used in the simulation study: Normal (0,1), Lognormal (0,1), Weibull (2,2), Gamma (2,2), Beta (4,4), and Exponential (4) (Erdoğan et al., 2019). These distributions have a wide range of supports, properties, and shapes. This simulation study examines the performance of four distinct methods

for estimating the parameter: the 3-degree Bernstein polynomial (BP3), the 3-degree rational Bernstein polynomial (RBP3), the 5-degree Bernstein polynomial (BP5), and the 5-degree rational Bernstein polynomial (RBP5). For each distribution, four different sample sizes are determined as n=20, 50, 100, and 200. For each given sample size, data are generated from the desired distributions, and the performance of four estimators is assessed by the mean squared error metric. The process is repeated N = 2000 times. We use the R programming language in the implementation phase. The optimization of the weights in the rational Bernstein polynomial estimator is carried out using the optim function in R with the L-BFGS-B algorithm, a quasi-Newton method for bound-constrained optimization. The least squares approach is employed for parameter estimation, where the objective is to minimize the squared differences between the estimated and theoretical cumulative distribution functions (CDFs). The optimization starts with initial parameter values based on the sample mean and standard deviation. All plots and tables presented in the study are generated directly from this simulation framework. Table 1 shows the obtained MSE values in the parameter estimation for each estimator. Simulation data shows that the MSE values for parameter estimation are usually lower for rational Bernstein polynomials (RBP3 and RBP5) compared to classical Bernstein polynomials (BP3 and BP5). This situation expresses how RBP lowers error levels by means of more weight factors and provides more modeling freedom. The simulation findings reveal that the values of mean squared error decrease with an increase in sample size. The parameter estimates based on rational estimators give very effective results with low MSE values even though in a small sample size. In Gamma and Beta distributions, both estimators based on Bernstein polynomials cannot give the desired result, and these estimators need to be increased in degree. Parameter estimates obtained using methods based on rational Bernstein polynomials have low MSE values without requiring high degrees. The estimator based on rational polynomials shows impressive performance, especially for Standard Normal, Lognormal, Weibull, and Exponential. The shape parameters and support sets of Gamma and Beta distributions do not work well with classical Bernstein polynomials, making them unsuitable for these distributions. Because the Gamma distribution is right-skewed and the Beta distribution only works within the [0,1] range, classical polynomials struggle to give accurate estimates for their parameters. These distributions need more flexible approaches; thus, rational Bernstein polynomials provide better results. Rational Bernstein polynomials overcome these limitations by better adapting to both the shape parameters and the characteristics of the support sets. This situation reveals that rational Bernstein polynomials are a reliable alternative in parameter estimation.

5. Real data example

This section of the study evaluates the performance of the two different methods using a real data set. The data represent the total milk production (in kilograms) at the first birth of 107 cows from the SINDI breed. This data set was initially reported by Cordeiro and Brito (2012). In a later study, Korkmaz et al. (2022) estimated the parameter of LEP distribution using several techniques with this data set.

The LEP distribution is defined by two positive parameters: α and β. If a random variable X follows the LEP distribution, denoted by $X \sim LEP(\alpha, \beta)$, its cumulative distribution function (CDF) is given by (Korkmaz et al., 2021):

$$F(X, \alpha, \beta) = e^{1-exp\{\alpha(-\log x)^{\beta}\}} \quad (8)$$

In this study, the parameters of LEP distribution are estimated using two different methods based on polynomials: the Bernstein polynomial (BP) and the rational Bernstein polynomial (RBP). We use polynomial degrees $m = 1, \ldots 10$ for estimating parameters using both methods. Table 2 displays the results of estimating parameters for the LEP distribution using BP and RBP methods with various degrees, as well as their Kolmogorov-Smirnov statistics. Figure 1 also shows how the estimated values of the parameters α and β of the LEP distribution change with the polynomial degree for both the BP and RBP methods. The results in Table 2 show that the parameter estimates obtained using the Bernstein polynomial and the rational Bernstein polynomial are different. Particularly at lower degrees, the Bernstein approach yields notably lower α and β estimates compared to RBP. However, as the degree increases, the estimates obtained by the Bernstein method begin to closely resemble those of RBP. The result indicates that while low-degree Bernstein polynomials have limited flexibility, higher degrees can approximate the flexibility provided by the rational Bernstein approach. To objectively identify the best-performing model, the Kolmogorov-Smirnov (KS) test was employed. Among the 20 models fitted, the rational Bernstein polynomial with degree $m = 2$ provided the highest p-value and the lowest KS statistics. As a result, the findings indicate that the RBP model with $m = 2$ offers a good balance between fit accuracy and model simplicity, making it the most appropriate choice in terms of statistical performance.

Table 1. Simulation results in terms of mean squared error for parameter estimation

Distribution	n	BP3	RBP3	BP5	RBP5
N(0, 1)	20	0.133223	0.050219	0.062688	0.054648
	50	0.109811	0.018841	0.035577	0.018530
	100	0.104897	0.009904	0.029363	0.009983
	200	0.102526	0.004931	0.026677	0.004801
Lognormal(0,1)	20	0.130477	0.050483	0.067375	0.049094
	50	0.115773	0.019922	0.051556	0.020106
	100	0.111698	0.010242	0.046587	0.009651
	200	0.110129	0.004777	0.043111	0.004836
Weibull(2, 2)	20	0.797413	0.067467	0.460105	0.157703
	50	0.741312	0.026374	0.442740	0.051945
	100	0.726775	0.015577	0.436631	0.026128
	200	0.721498	0.008929	0.433992	0.012952
Gamma(2, 2)	20	1.741081	0.605132	1.153526	0.940792
	50	1.680226	0.297883	1.133365	0.292105
	100	1.668933	0.149353	1.126791	0.142647
	200	1.647754	0.083783	1.106257	0.069236
Beta(4, 4)	20	6.730791	2.377447	4.582160	5.533315
	50	6.718733	1.149828	4.552348	1.416966
	100	6.695179	0.248238	4.540641	0.557964
	200	6.660357	0.247870	4.441913	0.231831
Exponential(4)	20	0.007695	0.007333	0.006575	0.007620
	50	0.004062	0.002065	0.002928	0.002351
	100	0.002992	0.001010	0.001514	0.000877
	200	0.002485	0.000455	0.001113	0.000451

Table 2. Parameter estimation for the LEP distribution using BP and RBP approaches with different degrees and Kolmogorov-Smirnov results.

Method	Degree	α	β	Kolmogorov-Smirnov	
				Statistics	P-value
BP	1	0.68357	0.74698	0.22392	0.00004
RBP	1	0.63461	0.78162	0.19222	0.00074
BP	2	0.65698	0.76678	0.20623	0.00022
RBP	2	0.79525	1.54467	0.10194	0.21613
BP	3	0.69611	0.90206	0.18451	0.00137
RBP	3	0.81065	1.60806	0.10396	0.19773
BP	4	0.68143	1.00445	0.15290	0.01343
RBP	4	0.81602	1.62943	0.10718	0.17106
BP	5	0.69684	1.06665	0.14346	0.02445
RBP	5	0.81554	1.62796	0.10693	0.17298
BP	6	0.70581	1.07350	0.14483	0.02246
RBP	6	0.81674	1.63252	0.10761	0.16767
BP	7	0.71575	1.16171	0.12807	0.05978
RBP	7	0.81701	1.63383	0.10778	0.16637
BP	8	0.71406	1.19822	0.11975	0.09291
RBP	8	0.81761	1.63591	0.10810	0.16393
BP	9	0.73477	1.20951	0.12340	0.07685
RBP	9	0.81896	1.64084	0.10883	0.15847
BP	10	0.73512	1.27025	0.11089	0.14382
RBP	10	0.81888	1.64048	0.10879	0.15883

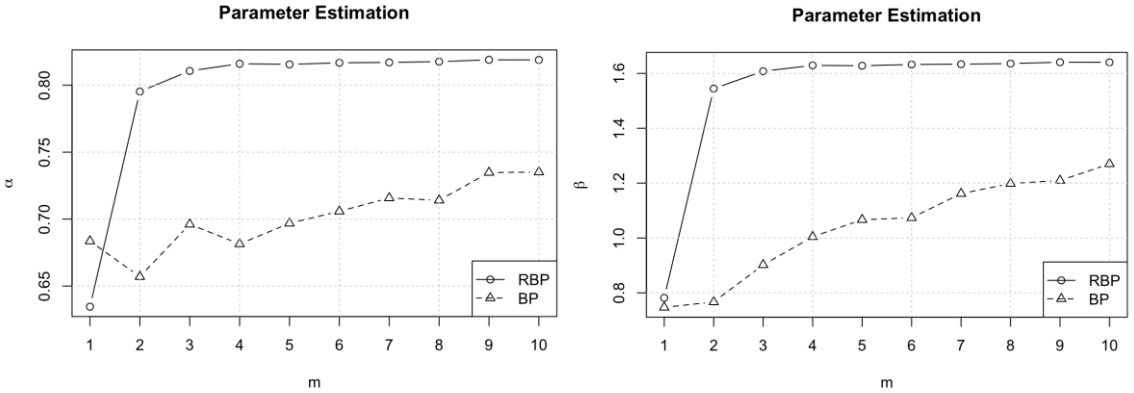


Figure 1. Estimations α and β for the First Birth Milk Production data of SINDI Cows

6. Conclusions

In conclusion, this study provides a comprehensive comparison of rational Bernstein polynomials (RBP) and Bernstein polynomials (BP) for parameter estimation across various distributions. By employing the least squares method, we demonstrated the ability of these estimators to accurately estimate the parameters of interest. Through a series of Monte Carlo simulations, we showed that RBP outperforms BP, particularly in terms of mean squared error (MSE), highlighting its superior accuracy and efficiency. Furthermore, we demonstrate the practical applicability of these methods by applying them to a real data set. The key contribution of this research lies in the introduction of RBP as a more flexible and adaptable tool for parameter estimation, offering distinct advantages over traditional BP estimators in complex data sets. The findings highlight that RBP, with its flexible and adaptable structure, provides a more reliable approach to parameter estimation, especially in the presence of complex and challenging data sets.

Declaration of Ethical Standards

The authors declare that they comply with all ethical standards.

Credit Authorship Contribution Statement

Author-1: Conceptualization, investigation, methodology and software, visualization and writing – original draft.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper

Data Availability Statement

All data generated or analyzed during this study are included in this published article.

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